MATHEMATICAL FOUNDATIONS OF DATA ANALYSIS

UNIT V

SINGULAR VALUE DE COMPOSITION OF A MATRIX

Definition: Singular Value Decomposition of a matrix:

A Singular Value Decomposition (SVD) of an m × n matrix A of rank r is a factorization **A = UΣV ^{T}** where **U and V are orthogonal** and **Σ =** $\begin{bmatrix} \boldsymbol{D} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}$ m×n in block form where **D = diag (d_1, d_2, ..., d_r)** where each $d_i > 0$, and $r \leq m$ and $r \leq n$.

Note1 : If A = UΣV ^{T} is any SVD for A as then:

1. r = rank A.
2. The numbers $d_1, d_2, ..., d_r$ are the singular values of A^TA in some order.

Note 2 :

Let A be a real m×n matrix. Then:

1. The eigen values of A^TA and AA^T are **real and non-negative.**
2. A^TA and AA^T have the same set of **positive eigen values**.

Definition: Singular values of the matrix A

Let A be a real m×n matrix. Let λ be an **eigenvalue of A^TA**, with non zero eigenvectors $q_i \in R^n$. Then the **real numbers** $\boldsymbol{\sigma_i} = \sqrt{\boldsymbol{\lambda_i}} = \|\boldsymbol{Aq_i}\|$ for i = 1, 2, ..., n, are called the **singular values of the matrix A** .

Definition: Singular matrix of A

Let A be a real, m×n matrix of rank r, with **positive singular values** $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ and $\sigma_i = 0$ if i > r. Define: **D_A = diag (σ_1, σ_2 ..., σ_r)** and $\Sigma_A = \begin{bmatrix} D_A & 0 \\ 0 & 0 \end{bmatrix}_{m \times n}$ Here Σ_A is in **block form** and is called the **Singular matrix of A.**

Definition: Two subspaces associated with a matrix A having m rows and n columns.

im A = { Ax | $x \in R^n$ } and col A = span {a | a is a column of A}.

Then **im A** is called the **image of A** (so named because of the linear transformation $R^n \rightarrow R^m$ with x → Ax); and **col A** is called the **column space of A**.

Note : **im A = col A.**

Definition: Singular Value Decomposition (SVD) of A

Definition: Let A be a real m×n matrix, and let $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ be the positive **singular values of A**. Then **r is the rank of A** and we have the factorization **$A = P\Sigma_A Q^T$** where **P and Q are orthogonal matrices.** The factorization **$A = P\Sigma_A Q^T$,** where **P and Q are orthogonal matrices**, is called a **Singular Value Decomposition (SVD) of A.** This decomposition is not unique.

Reference:

https://math.emory.edu/~lchen41/teaching/2020_Fall/Section_8-6.pdf